The scattering of gravity waves by turbulence

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(Received 17 June 1958)

A theory is developed to describe the properties of waves on the free surface of a liquid in turbulent motion. The distinction between the interacting wave and turbulent motions is achieved by separating the velocity field uniquely into a surface-induced contribution characteristic of the wave motion and a vorticity-induced contribution associated with the turbulence. The theory is applied to the scattering of gravity waves passing over the surface of deep water which has a turbulent motion of sufficiently low mean square vorticity. Expressions are derived for the directional distribution of the scattered wave (equation 4.19) and for the logarithmic decrement resulting from the scattering (4.20). It is shown that, under typical conditions in open sea, the attenuation from scattering will be greater than that from direct viscous dissipation for wavelengths greater than about 3 m.

1. Introduction

The problem of wave motion on the surface of a turbulent fluid has several different aspects. One of these is concerned with the propagation of an incident gravity wave through a region of turbulent fluid (water) and with the attenuation of the wave that results from the interation between the turbulence and wave motion. This aspect may well be of interest in oceanography, since there is little doubt that the motion in the upper layers of the open ocean is usually turbulent, although the turbulent velocity fluctuations are likely to be much smaller than those generally found in the atmosphere. In studies of propagation of surface waves over great distances across the ocean, it is probable that the scattering of waves by oceanic turbulence may have a significant effect. For example,† if waves originating in one storm have to pass through another storm (where presumably the turbulent intensity is high) to reach an observer, one effect of scattering is to increase the average path length taken by wave energy in passing from source to observer, and so to distribute and delay their arrival. Such a delay has been observed by Darbyshire (1952), and the most natural explanation seems to be in terms of this scattering process.

A second aspect of the interaction between surface waves and turbulence in the water is concerned with the generation of waves on the surface of the liquid, and with the derivation of relations between the properties of the surface displacements and the structure of the turbulence that induces them. An associated

† I am indebted to the referee for pointing out this example.

problem concerns the rate at which energy is transferred from the turbulence to the waves and is carried away by the wave motion from, say, a localized region of turbulence. It might be anticipated from visual observations of the water surface below, say, the spillway of a dam, that the transfer of energy from turbulence to propagating waves is relatively slow, so that the process would be relatively insignificant in most oceanographical contexts, although it may be important in other situations. This second aspect of the problem will be discussed in a succeeding paper.

Let us return to consideration of the problem of present interest, namely, the attenuation of an incident wave moving across the surface of a deep turbulent liquid, where the intensity of the turbulence is sufficiently small that the process of wave generation is unimportant. One can see physically that there are two possible types of interaction, each of which will result in an attenuation of the incident wave. The first can be called an 'eddy viscosity interaction' in which energy is transferred from the wave motion to the turbulence. The passage of the wave results in straining the elements of fluid near the surface in a manner which is almost, but not quite, periodic in time. The mean strain per cycle of the incident wave is of second order, namely $(\alpha/\lambda)^2$, where α is the amplitude and λ the wavelength of the incident wave. The wave motion therefore provides a mechanism for stretching the vortex lines that operates in addition to the stretching inherent in the turbulence itself, and so tends to increase $\overline{\omega^2}$, the mean square vorticity associated with the turbulence, until a balance is reached with the diffusive action of viscosity. Two comments are pertinent to this eddy viscosity type of interaction. The first is that the energy of the waves is transferred to turbulence having a length scale smaller than the wavelength of the incident wave and so, from the point of view of the turbulence, provides a source of energy at fairly large wave numbers (though not as large as those associated with the dissipation range). In the second place, the additional straining process is, for a given level of $\overline{\omega^2}$ in the water, of second order in α/λ and so it might be expected to be most important for a given wavelength when the amplitude α of the incident wave is large and less important when α is small, as in ocean swell.

Another process which might be expected to influence the propagation of surface waves can be described as a scattering phenomenon. The presence of random velocity fluctuations in the water of a length scale comparable with the wavelength of the waves will result in the convective distortion of the wave fronts, and so in the establishment of a scattered wave field. It is clear that the amplitude of the scattered wave system will be proportional to the amplitude of the incident wave, as in other scattering problems (see, for example, the discussion by Batchelor (1957), which contains a bibliography of significant earlier work in these fields) so that the incident wave suffers a true logarithmic decrement. This scattering effect is clearly of lower order in (α/λ), and the attenuation time will be independent of wave amplitude. Furthermore, since the incident wave will be supposed to have a very small slope, only terms of lowest order in α/λ will be retained in the analysis, so that only the scattering phenomenon would be expected to be described. This makes it possible to neglect the effect of the waves on the turbulence, so that the turbulent field can be regarded as prescribed by other considerations, such as the existence of mean velocity gradients set up by ocean currents.

One of the fundamental differences between the scattering of gravity waves and most other scattering problems that have been considered is that gravity waves in deep water are dispersive: the phase velocity is a function of wavelength. This suggests that from the outset, the analysis might be developed most conveniently in terms of the Fourier components of both the incident wave field and the vorticity fluctuations characteristic of the turbulence. This plan is adopted in the later sections of this paper.

2. Specification of the problem

The first difficulty with which we are faced in discussing the wave motion on the surface of a turbulent liquid is that we can define neither a velocity potential nor a stream function, since the turbulence is essentially rotational and three-dimensional, and consequently we are denied many of the simple and powerful methods developed in the past for the theory of water waves. This difficulty is overcome to some extent by expressing the Navier–Stokes equations for the motion of the water, namely,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\,.\,\nabla\mathbf{u} = -\nabla(p/\rho - gz) + \nu\nabla^2\mathbf{u} \tag{2.1}$$

in the equivalent form

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u}\times\boldsymbol{\omega} = -\nabla(p/\rho - gz + \tfrac{1}{2}u^2) - \nu\nabla\times\boldsymbol{\omega}, \tag{2.2}$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ is the vorticity at the point $\mathbf{x}$ and the z co-ordinate axis is taken vertically downwards. In regions where the motion is irrotational, that is $\boldsymbol{\omega} = 0$, equation (2.2) reduces to the Bernoulli equation, and the combination

$$(p/\rho - gz + \tfrac{1}{2}u^2)$$

forms an acceleration potential for the motion. In the turbulent regions $\boldsymbol{\omega} \neq 0$, and the two additional terms of (2.2) describe the effects of viscous damping and of the interactions between the waves and the turbulence and the turbulence with itself.

This immediately raises a second difficulty which concerns the separation of the motion into components associated with the waves and the turbulence respectively, since it is the interaction between these two components that we wish to study. As far as the vorticity field is concerned there is little problem, since in a fluid of small viscosity it is almost entirely associated with the turbulence. But for a velocity field that does not vanish at infinity in all directions, there is no simple unambiguous separation into rotational and irrotational parts, and we are obliged to seek an alternative representation. In the present problem, it is convenient to perform the separation in the following manner.

Since

$$\boldsymbol{\omega} = \nabla\times\mathbf{u},$$

and the liquid can be assumed to be incompressible, so that $\nabla . \mathbf{u} = 0$, it follows that

$$\nabla^2 \mathbf{u} = -\nabla \times \boldsymbol{\omega}. \tag{2.3}$$

The solution of (2.3) for the velocity components $\mathbf{u}$ can be expressed as

$$\mathbf{u}(\mathbf{x}) = \frac{1}{4\pi}\int_V \nabla \times \boldsymbol{\omega}(\mathbf{y}) \frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{4\pi}\int_S \{|\mathbf{x}-\mathbf{y}|^{-1} d\mathbf{S} . \nabla \mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{y}) \nabla |\mathbf{x}-\mathbf{y}|^{-1} . d\mathbf{S}\}, \tag{2.4}$$

where the first integral is taken over the domain occupied by the fluid and the second over the surface bounding this domain. Several comments concerning this representation are worthy of attention. The first is that it clearly provides a unique separation of the velocity field into components associated with a given vorticity field and with a given velocity distribution over the boundaries of the domain. In the present problem, this corresponds to contributions to the velocity field from the turbulence and from the wave motion of the surface. Secondly, the separation is *not* one into rotational and irrotational parts of the velocity field, but into parts $\mathbf{u}_1$ and $\mathbf{u}_2$ which satisfy the equations $\nabla^2\mathbf{u}_1 = -\nabla \times \boldsymbol{\omega}$ and $\nabla^2 \mathbf{u}_2 = 0$.

The distinction may be illustrated clearly in some simple examples. The velocity field associated with a localized region of vorticity in an unconfined fluid at rest at infinity is given entirely by the first (or vorticity) integral of (2.4) even outside the vortical region where the motion is irrotational. Again, if our control surface is taken in the interior of a fluid in uniform shearing motion, then the flow inside the surface is described wholly by the second (or surface) integral, since the vorticity is uniform and $\nabla \times \boldsymbol{\omega} = 0$. This second example also illustrates one restriction on the usefulness of this representation (though not, of course, on its validity). If the control surface is chosen arbitrarily in the fluid, then the motion over these surfaces will depend in general on the vorticity distribution, so that the two parts in this representation are kinematically related, and little is achieved by it. However, if the control surfaces are chosen to coincide with physical boundaries of the flow, the surface motion can be prescribed and the vorticity distribution is related to it only through the dynamical interaction. The two contributions to the velocity field represented by the terms on the right of (2.4) can conveniently be called the vortex-induced and surface-induced contributions to the velocity field.

The simplest scattering problem, analogous to the 'single scattering' of sound or radio waves, occurs when a surface wave traverses a region where the mean square vorticity fluctuations are sufficiently small that in the interaction term $\mathbf{u} \times \boldsymbol{\omega}$ of equation (2.2), the velocity $\mathbf{u}$ near the surface is determined predominantly by the surface-induced contribution from the incident wave. It appears later that the influence of the turbulence on the wave motion decreases very rapidly with increasing depth, so that we need only be concerned with the restrictions imposed by this condition when applied to the motion near the surface. The mean square value of the first integral is

$$\frac{1}{16\pi^2}\iint \overline{\nabla \times \boldsymbol{\omega}(\mathbf{y}) . \nabla \times \boldsymbol{\omega}(\mathbf{z})} \frac{d\mathbf{y}\, d\mathbf{z}}{yz},$$

where the origin is chosen at $\mathbf{x} = 0$, and if the length scale of the inhomogeneities in the vorticity field is much greater than the integral length scales of the vorticity covariances, it can be shown that this expression is of order

$$\frac{\overline{\omega^2}}{l_\omega^2} L_\omega^4,$$

where l_ω and L_ω are differential and integral length scales of the vorticity fluctuations. The root mean square magnitude of the first integral in (2.4) is then of order $(\overline{\omega^2})^{\frac{1}{2}} L_\omega^2/l_\omega$. Near the surface, the second integral is of order αn, where α represents the height of the incident waves and n their frequency, which for gravity waves of wavelength λ is of order $(g/\lambda)^{\frac{1}{2}}$. A necessary condition, therefore, for the validity of a first-order scattering theory is that

$$u_t = \frac{(\overline{\omega^2})^{\frac{1}{2}} L_\omega^2}{l_\omega} \ll \frac{\alpha g^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}},$$

where u_t represents the root-mean-square turbulent velocity fluctuations.

This restriction relates the Froude number of the turbulence to length scales of the turbulence and of the waves, since it can be expressed as

$$\left(\frac{L_\omega}{\lambda}\right)^{\frac{1}{2}} F^{\frac{1}{2}} \ll \frac{\alpha}{\lambda}, \tag{2.5}$$

where $F = \mathbf{u}_t^2/gL_\omega$. Since, in a linearized wave theory α/λ itself must be small, of order $1/10$ or less, we require that $F^{\frac{1}{2}}(L_\omega/\lambda)^{\frac{1}{2}}$ be even smaller. An alternative expression of (2.5), using the condition $\alpha/\lambda \ll 1$, is that

$$u_t \lll (g\lambda)^{\frac{1}{2}} = c(\kappa), \tag{2.6}$$

the phase velocity of the incident waves, where the triple inequality sign can be interpreted to mean 'is less by a factor of probably one hundred than'.

3. The wave propagation equation

The process of taking the divergence of (2.2) yields

$$\nabla^2(p/\rho - gz + \tfrac{1}{2}u^2) = \nabla . \mathbf{u} \times \boldsymbol{\omega}. \tag{3.1}$$

This equation provides the starting-point of the derivation of an inhomogeneous wave equation that describes the propagation of surface waves on deep turbulent water. There are three main steps in the analysis:

(*a*) Consider $(p/\rho - gz + \frac{1}{2}u^2) = f$, say, as our basic dependent variable and solve (3.1) formally for f in terms of its normal gradient $\partial f/\partial z$ at the equilibrium level $z = 0$, and of the distribution of $\nabla . \mathbf{u} \times \boldsymbol{\omega}$.

(*b*) Express $\partial f/\partial z$ at $z = 0$ in terms of the surface acceleration and the vorticity distribution by means of the vertical component of the momentum equation (2.1). These two steps give the function f throughout the fluid in terms of the surface acceleration and the vorticity distribution.

(*c*) The function f at the plane $z = 0$ is also expressed in terms of the surface displacement, using the boundary condition of constant pressure. This leads to the required equation between the surface displacement, surface acceleration and the vorticity distribution.

It will be supposed that the motion in the water is statistically homogeneous in planes parallel to the undisturbed surface $z = 0$, both on the grounds of analytical convenience and in the belief that this will represent a good approximation to the oceanographical situation. The motion can then be described by stationary random functions of the position variables x and y parallel to this plane (z being taken vertically downwards) and we can define Fourier–Stieltjes transforms $d\phi(\boldsymbol{\kappa}, z, t)$ and $d\chi(\boldsymbol{\kappa}, z, t)$ of the functions occurring in equation (3.1):

$$p/\rho - gz + \tfrac{1}{2}u^2 = \int d\phi(\boldsymbol{\kappa}, z, t)\, e^{i\boldsymbol{\kappa}.\mathbf{x}}, \tag{3.2}$$

$$\nabla . \mathbf{u} \times \boldsymbol{\omega} = \int d\chi(\boldsymbol{\kappa}, z, t)\, e^{i\boldsymbol{\kappa}.\mathbf{x}}, \tag{3.3}$$

where $\boldsymbol{\kappa} = (\kappa_1, \kappa_2)$ represents a two-dimensional horizontal wave-number so that $\boldsymbol{\kappa}.\mathbf{x} = \kappa_1 x + \kappa_2 y$, and the integrals are taken over all values of $\boldsymbol{\kappa}$. The analysis below is divided into three sections corresponding to the three main steps that have already been described.

(*a*) *Formal solution of equation* (3.1)

The relation between the Fourier–Stieltjes transforms $d\phi(\boldsymbol{\kappa}, z, t)$ and $d\chi(\boldsymbol{\kappa}, z, t)$ equivalent to (3.1) is

$$\left(\frac{\partial^2}{\partial z^2} - \kappa^2\right) d\phi(\boldsymbol{\kappa}, z) = d\chi(\boldsymbol{\kappa}, z), \tag{3.4}$$

where $\kappa^2 = \kappa_1^2 + \kappa_2^2$ and the time dependence is implicit. The boundary conditions to be imposed on (3.1) are that $\partial f/\partial z$ at the equilibrium level $z = 0$ is supposed to be given (it will be related to the surface acceleration in step (*b*)), and that both f and $\nabla . \mathbf{u} \times \boldsymbol{\omega}$ remain finite as $z \to \infty$, since the turbulent energy is presumed to be finite at very great depths. The corresponding conditions on the functions $d\phi$ and $d\chi$ in equation (3.4) are

$$\left.\begin{aligned} &\frac{\partial}{\partial z} d\phi(\boldsymbol{\kappa}, z, t) = d\Phi(\boldsymbol{\kappa}, t), \quad \text{say,} \quad \text{at } z = 0, \\ &d\phi(\boldsymbol{\kappa}, z, t), \quad d\chi(\boldsymbol{\kappa}, z, t) \quad \text{finite,} \quad \text{as } z \to \infty. \end{aligned}\right\} \tag{3.5}$$

The solution to (3.4) can be expressed as

$$d\phi(\boldsymbol{\kappa}, z, t) = \frac{1}{2\kappa}\int_0^z d\chi(\eta)\,\{e^{-\kappa(z-\eta)} - e^{\kappa(z-\eta)}\}\, d\eta + d\alpha\, e^{\kappa z} + d\beta\, e^{-\kappa z}, \tag{3.6}$$

where $\kappa = |\boldsymbol{\kappa}|$, η is a variable of integration in the vertical direction, and the dependance of $d\chi$ upon $\boldsymbol{\kappa}$ and time t is regarded as implicit. The functions $d\alpha(\boldsymbol{\kappa}, t)$ and $d\beta(\boldsymbol{\kappa}, t)$ are to be determined from the conditions (3.5). As $z \to \infty$, we find from (3.6) that

$$d\phi(z) \to e^{\kappa z}\left\{d\alpha - \frac{1}{2\kappa}\int_0^\infty d\chi(\eta)\, e^{-\kappa\eta}\, d\eta\right\} + \frac{1}{2\kappa}\int_0^z d\chi(\eta)\, e^{-\kappa(z-\eta)}\, d\eta,$$

so that in order to satisfy the condition of boundedness of $d\phi(z)$ as $z \to \infty$, it is necessary that

$$d\alpha(\boldsymbol{\kappa}, t) = \frac{1}{2\kappa}\int_0^\infty d\chi(\boldsymbol{\kappa}, \eta, t)\, e^{-\kappa\eta}\, d\eta. \tag{3.7}$$

Note that if the vorticity fluctuations extend to only a finite depth, then in fact $d\phi(\boldsymbol{\kappa}, z, t) \to 0$ as $z \to \infty$; and the fluid is at rest at very great depths.

Using the relation (3.7), the solution (3.6) can now be expressed as

$$d\phi(\boldsymbol{\kappa}, z, t) = \frac{1}{2\kappa}\int_0^z d\chi(\eta)\, e^{-\kappa(z-\eta)}\, d\eta + \frac{1}{2\kappa}\int_z^\infty d\chi(\eta)\, e^{\kappa(z-\eta)}\, d\eta + d\beta\, e^{-\kappa z},$$

so that $$\left[\frac{\partial}{\partial z} d\phi(\boldsymbol{\kappa}, z, t)\right]_{z=0} = \frac{1}{2}\int_0^\infty d\chi(\eta)\, e^{-\kappa\eta}\, d\eta - \kappa\, d\beta = d\Phi(\boldsymbol{\kappa}, t), \quad \text{say},$$

the boundary function in which we desire to express our solution. Thus

$$d\beta(\boldsymbol{\kappa}, t) = \frac{1}{2\kappa}\int_0^\infty d\chi(\boldsymbol{\kappa}, \eta, t)\, e^{-\kappa\eta}\, d\eta - \frac{1}{\kappa} d\Phi(\boldsymbol{\kappa}, t), \tag{3.8}$$

and the solution (3.6) finally becomes

$$d\phi(\boldsymbol{\kappa}, z, t) = \frac{1}{2\kappa}\int_0^z d\chi(\eta)\,\{e^{-\kappa(z-\eta)} - e^{\kappa(z-\eta)}\}\, d\eta + \frac{1}{\kappa}\cosh \kappa z \int_0^\infty d\chi(\eta)\, e^{-\kappa\eta}\, d\eta - \frac{e^{-\kappa z}}{\kappa} d\Phi(\boldsymbol{\kappa}, t). \tag{3.9}$$

(*b*) *Relation between $d\Phi$ and the surface acceleration*

This next step is achieved by considering the vertical component of the vector equation (2.2) at the surface. If the suffix z is taken to denote downwards vertical components, then

$$\frac{\partial}{\partial z}(p/\rho - gz + \tfrac{1}{2}u^2) = (\mathbf{u} \times \boldsymbol{\omega} - \nu\nabla \times \boldsymbol{\omega})_z - \frac{\partial u_z}{\partial t}, \tag{3.10}$$

or if $\xi(x, y, t)$ represents the surface displacement measured upwards,

$$\frac{\partial}{\partial z}(p/\rho - gz + \tfrac{1}{2}u^2) = (\mathbf{u} \times \boldsymbol{\omega} - \nu\nabla \times \boldsymbol{\omega})_z + \frac{\partial^2 \xi}{\partial t^2}, \tag{3.11}$$

if the surface slope is small.

To express this equation in terms of its Fourier components, we must now introduce the further Fourier–Stieltjes transforms

$$\left.\begin{aligned} \mathbf{u} \times \boldsymbol{\omega} &= \int d\boldsymbol{\Gamma}(\boldsymbol{\kappa}, z, t)\, e^{i\boldsymbol{\kappa}.\mathbf{x}}, \\ (\nabla \times \boldsymbol{\omega})_z &= \int d\Delta(\boldsymbol{\kappa}, z, t)\, e^{i\boldsymbol{\kappa}.\mathbf{x}}, \end{aligned}\right\} \tag{3.12}$$

and for the surface displacement

$$\xi(x, y, t) = \int dA(\boldsymbol{\kappa}, t)\, e^{i\boldsymbol{\kappa}.\mathbf{x}}. \tag{3.13}$$

It is clear that $d\boldsymbol{\Gamma}$ is simply related to the function defined by (3.3), for from the first of (3.12),

$$\begin{aligned} \nabla . \mathbf{u} \times \boldsymbol{\omega} &= \int\left(i\boldsymbol{\kappa} . d\boldsymbol{\Gamma} + \frac{\partial}{\partial z} d\boldsymbol{\Gamma}_z\right) e^{i\boldsymbol{\kappa}.\mathbf{x}} \\ &= \int d\chi\, e^{i\boldsymbol{\kappa}.\mathbf{x}}, \end{aligned}$$

so that

$$d\chi(\boldsymbol{\kappa}, z, t) = i\boldsymbol{\kappa} . d\boldsymbol{\Gamma}(\boldsymbol{\kappa}, z, t) + \frac{\partial}{\partial z} d\Gamma_z(\boldsymbol{\kappa}, z, t). \tag{3.14}$$

Equation (3.11) can therefore be expressed as a relation between these Fourier–Stieltjes transforms:

$$\left[\frac{\partial}{\partial z}\,d\phi\right]_{z=0} = [d\Gamma_z - \nu\,d\Delta]_{z=0} + \frac{\partial^2}{\partial t^2}dA$$
$$= d\Phi,$$

from (3.5), where again the dependence of the various functions on $\boldsymbol{\kappa}$, t and, where relevant, z, is understood. Equation (3.9) therefore becomes

$$d\phi(\boldsymbol{\kappa}, z, t) = \frac{1}{2\kappa}\int_0^z d\chi(\eta)\,\{e^{-\kappa(z-\eta)} - e^{\kappa(z-\eta)}\}\,d\eta + \kappa^{-1}\cosh\kappa z\int_0^\infty d\chi(\eta)\,e^{-\kappa\eta}\,d\eta$$
$$-\kappa^{-1}\,e^{-\kappa z}\left\{d\Gamma_z(\boldsymbol{\kappa}, t) - \nu\,d\Delta(\boldsymbol{\kappa}, t) + \frac{\partial^2}{\partial t^2}dA(\boldsymbol{\kappa}, t)\right\}, \quad (3.15)$$

where the quantities $d\Gamma_z$ and $d\Delta$ are taken at $z = 0$.

This equation provides a relation (admittedly rather a complicated one) between the Fourier–Stieltjes components of the function $f = p/\rho - gz + \frac{1}{2}u^2$ and those involving the vorticity distribution and the surface accelerations. It is subject only to the restriction that the surface gradients are small; no approximations have yet been made concerning the relative magnitudes of velocities associated with the waves and the turbulence. It is therefore applicable equally to the problem of scattering of gravity waves by turbulence of low intensity and to the problem of wave *generation* when the turbulent intensity is greater. The further consideration of this latter problem is postponed to a later paper; from this point in the present paper attention will be confined to the scattering problem in which the turbulent intensity is sufficiently small that the condition (2.5) is satisfied.

(*c*) *Derivation of the inhomogeneous wave equation*

The boundary condition at the free surface $z = -\xi$ is that the pressure is constant and can be taken as zero. Under the condition (2.5), the velocity near the surface is predominantly a result of the surface-induced contributions, so that $\frac{1}{2}u^2$ is of order $\alpha^2 n^2$, where α is the height of the incident wave and n is its frequency. For gravity waves in deep water, n^2 is of order g/λ, where λ is the wavelength, so that $\frac{1}{2}u^2$ is of order $g\alpha^2/\lambda$. To the first order in the wave slope α/λ, therefore,

$$[p/\rho - gz + \tfrac{1}{2}u^2]_{z=0} \simeq [p/\rho - gz + \tfrac{1}{2}u^2]_{z=-\xi}$$
$$\simeq g\xi.$$

The free surface boundary condition, together with (2.6) requiring α/λ to be small, therefore leads to the relation

$$g\,dA(\boldsymbol{\kappa}, t) = [d\phi(\boldsymbol{\kappa}, z, t)]_{z=0} \quad (3.16)$$

between the Fourier–Stieltjes transforms defined by (3.2) and (3.13). This, together with (3.15), provides an equation for the propagation of the components $dA(\boldsymbol{\kappa}, t)$ of the surface displacement

$$\left(\frac{\partial^2}{\partial t^2} + g\kappa\right)dA(\boldsymbol{\kappa}, t) = \int_0^\infty d\chi(\eta)\,e^{-\kappa\eta}\,d\eta - d\Gamma_z(\boldsymbol{\kappa}, t) - \nu\,d\Delta(\boldsymbol{\kappa}, t). \quad (3.17)$$

The relation (3.14) can be used to express $d\chi(\eta)$ in terms of $d\mathbf{\Gamma}(\eta)$, and on substitution into (3.17) and integration by parts, we find that

$$\left\{\frac{\partial^2}{\partial t^2}+\left(\frac{g}{\kappa}\right)\kappa^2\right\} dA(\boldsymbol{\kappa},t) = \int_0^\infty (\kappa\, d\Gamma_z + i\boldsymbol{\kappa}\,.\,d\mathbf{\Gamma})\, e^{-\kappa\eta}\, d\eta - \nu\, d\Delta(\boldsymbol{\kappa},t), \qquad (3.18)$$

where, yet again, the variation of $d\mathbf{\Gamma}$ with $\boldsymbol{\kappa}$, η and time t is to be understood and $d\Delta$ is taken at $z = 0$.

The left-hand side of this equation is recognizable as the Fourier transform of a wave equation in which the phase velocity of the waves is $(g/\kappa)^{\frac{1}{2}}$. The right-hand side represents the influence of the inhomogeneities introduced by the turbulence, together with the viscous damping term. It will be noticed that, in view of the exponential factor in the integral, the vorticity fluctuations at depths greater than about a wavelength have little influence upon the propagation of surface waves, verifying the statement made in anticipation in § 2. The viscous dissipation term is probably negligible for all but very short waves. In the absence of turbulent scattering, an expression given by Lamb (1932, p. 624) indicates that a water wave of wavelength 20 m travels a distance of 8400 km before its amplitude is reduced by a factor e^{-1}. It is therefore likely (an *a posteriori* justification can be made from the results of the next section) that unless the mean square turbulent vorticity fluctuations in the ocean are exceedingly minute, the effect of viscous dissipation in waves of moderate wavelength will be small compared to the effects of turbulent scattering, and the term $\nu\, d\Delta$ in (3.18) can be neglected.

4. The scattered wave system

Suppose that, at an initial instant, a wave of amplitude α and wavelength $2\pi/k$ is travelling along the surface of the water in the direction of the positive x-axis. If the fluctuations in vorticity in the water are sufficiently small, the motion is, to a first approximation, that of an irrotational surface wave in which the surface displacement is

$$\alpha e^{-i(kx-nt)},$$

and the surface-induced components of the velocity field are

$$\left.\begin{aligned} u_1 &= \alpha n\, e^{-i(kx-nt)}\, e^{-kz}, \\ u_2 &= 0, \\ u_3 &= -i\alpha n\, e^{-i(kx-nt)}\, e^{-kz}, \end{aligned}\right\} \qquad (4.1)$$

where the suffices 1, 2 and 3 indicate velocity components in the x, y and z directions respectively, and the wave frequency $n = (gk)^{\frac{1}{2}}$.

The next approximation, which describes the scattered wave field is obtained by substituting from (4.1) into the interaction term of the inhomogeneous equation (3.18). To do this, we make use of the theorem which states that if $f(\boldsymbol{\kappa})$ is the Fourier transform of $F(\mathbf{x})$, then the Fourier transform of $F(\mathbf{x})\, e^{-ikx}$ is

$$\begin{aligned}\int F(\mathbf{x})\, e^{-ikx}\, e^{i\boldsymbol{\kappa}.\mathbf{x}}\, d\mathbf{x} &= \int F(\mathbf{x})\, e^{i(\kappa_1-k)\,x+i\kappa_2 y}\, d\mathbf{x} \\ &= f(\mathbf{K}), \end{aligned} \qquad (4.2)$$

where the new wave-number vector $\mathbf{K} = (\kappa_1 - k, \kappa_2)$. Now $d\Gamma_1$ is the Fourier–Stieltjes transform of $(u_2\omega_3 - u_3\omega_2)$ or of $-u_3\omega_2$ approximately, since from (4.1), to the first approximation, the surface-induced contribution to u_2 from the incident wave vanishes. Therefore, if the vorticity fluctuations are represented by

$$\boldsymbol{\omega}(\mathbf{x}, t) = \int d\boldsymbol{\Omega}(\boldsymbol{\kappa}, z, t)\, e^{i\boldsymbol{\kappa}.\mathbf{x}}, \tag{4.3}$$

where $\boldsymbol{\kappa} = (\kappa_1, \kappa_2)$, we have from (4.1) and (4.2) that

$$d\Gamma_1 = i\alpha n\, e^{int}\, e^{-kz} d\Omega_2(\mathbf{K}).$$

Similarly,

$$d\Gamma_2 = -i\alpha n\, e^{int}\, e^{-kz}\{d\Omega_1(\mathbf{K}) + i\, d\Omega_3(\mathbf{K})\},$$

$$d\Gamma_3 = -i\alpha n\, e^{int}\, e^{-kz} d\Omega_2(\mathbf{K}),$$

so that from equation (3.18),

$$\left(\frac{\partial^2}{\partial t^2} + g\kappa\right) dA(\boldsymbol{\kappa}, t) = \alpha n\, e^{int}\, d\psi(\mathbf{K}, \boldsymbol{\kappa}, t), \tag{4.4}$$

say, where

$$d\psi(\mathbf{K}, \boldsymbol{\kappa}, t) = \int_0^\infty \{\kappa_2\, d\Omega_1(\mathbf{K}) - (\kappa_1 + i\kappa)\, d\Omega_2(\mathbf{K}) + i\kappa_2\, d\Omega_3(\mathbf{K})\}\, e^{-(\kappa+k)\eta}\, d\eta, \tag{4.5}$$

the dependence of $d\Omega_i(\mathbf{K})$ upon η and time t being understood.

Equation (4.4) specifies the rate of growth of the wavelets of wave-number $\boldsymbol{\kappa}$ set up by the passage of the wave of wave-number k through the turbulent fluid, where the subsequent scattering of these wavelets themselves is neglected. This process is analogous to that of 'single scattering' of sound or radio waves in a medium in which the wave velocity varies irregularly from one point to another. In the case of gravity waves on the surface of a liquid, however, the phase and group velocities are determined not by the local surface conditions but by the motion throughout a whole layer of fluid near the surface, and this fact is reflected by the integral expression as given by (4.5) that appears on the right-hand side of the scattering equation (4.4).

We will suppose that at the initial instant $t = 0$ the amplitude of the scattered wave system is zero, and we will investigate the growth of the scattered components during subsequent times. The initial conditions relevant to (4.4) are then that

$$dA(\boldsymbol{\kappa}, t) = \frac{\partial}{\partial t} dA(\boldsymbol{\kappa}, t) = 0 \quad \text{at} \quad t = 0,$$

so that the solution can be expressed as

$$\begin{aligned} dA(\boldsymbol{\kappa}, t) &= \frac{n}{2in'} \int_0^t \alpha\, d\psi(\mathbf{K}, \boldsymbol{\kappa}, \tau)\, e^{in\tau}\{e^{in'(t-\tau)} - e^{-in'(t-\tau)}\}\, d\tau \\ &= \frac{n}{2in'} \int_0^t \alpha\, d\psi(\mathbf{K}, \boldsymbol{\kappa}, \tau)\, \{e^{in't}\, e^{i(n-n')\tau} - e^{-in't}\, e^{i(n+n')\tau}\}\, d\tau, \end{aligned} \tag{4.6}$$

where $n = (gk)^{\frac{1}{2}}$ and $n' = (g\kappa)^{\frac{1}{2}}$. The variation with time of $d\psi(t)$ is determined by the variation of the vorticity fluctuations with time, and in many problems of interest, particularly in oceanographical contexts, this is very much slower than the time variations characteristic of the waves. It is possible to neglect the temporal variation of the vorticity if the time taken for the wave to propagate through a distance equal to the length scale L_ω of the vorticity fluctuations is small compared with the time scale of these fluctuations, that is, if

$$\frac{L_\omega}{(g\lambda)^{\frac{1}{2}}} \ll (\overline{\omega^2})^{\frac{1}{2}},$$

where λ is the wavelength. This condition can be restated in terms of the Froude number of the turbulence as follows:

$$\left(\frac{L_\omega}{\lambda}\right)^{\frac{1}{2}} F^{\frac{1}{2}} \ll \frac{L_\omega}{l_\omega},$$

where F is the Froude number and l_ω the differential length scale defined at the end of §2. In general the integral scale L_ω would be expected to be rather larger than the differential scale l_ω so that this condition is much less restrictive than (2.5) and should hold automatically when (2.5) is satisfied.

The amplitude α of the incident wave also is contained within the time integral of (4.6). Its variation with time occurs as a result of the loss of energy to the scattered components, but since the vorticity level in the water has already been presumed small, it can be anticipated that this variation is also likely to be slow. For times less than the time scale γ^{-1} of the attenuation of the incident wave due to scattering, therefore, equation (4.6) simplifies to

$$dA(\boldsymbol{\kappa}, t) \doteq \frac{\alpha n}{2in'} d\psi(\mathbf{K}, \boldsymbol{\kappa}, 0) \int_0^t \{e^{in't} e^{i(n-n')\tau} - e^{-in't} e^{i(n+n')\tau}\} d\tau,$$

$$= \frac{-\alpha n}{2n'} d\psi(\mathbf{K}, \boldsymbol{\kappa}) \left\{\frac{e^{int} - e^{in't}}{n-n'} - \frac{e^{int} - e^{-in't}}{n+n'}\right\}, \tag{4.7}$$

where α and $d\psi$ represent the appropriate quantities at the initial instant $t = 0$. This equation can be expressed as

$$dA(\boldsymbol{\kappa}, t) = \frac{-\alpha n}{2n'} d\psi(\mathbf{K}, \boldsymbol{\kappa})\, t \left\{\frac{e^{i\sigma} - e^{i\sigma'}}{\sigma - \sigma'} - \frac{e^{i\sigma} - e^{-i\sigma'}}{\sigma + \sigma'}\right\},$$

where $\sigma = nt = (gk)^{\frac{1}{2}} t$ and $\sigma' = n't = (g\kappa)^{\frac{1}{2}} t$. The spectrum of the scattered waves is

$$\Phi(\boldsymbol{\kappa}, t) = \frac{\overline{dA(\boldsymbol{\kappa}, t)\, dA^*(\boldsymbol{\kappa}, t)}}{d\kappa_1\, d\kappa_2}$$

$$= \frac{\alpha^2 n^2}{4n'^2} \Psi(\mathbf{K}, k)\, t^2 \Gamma(\kappa, k, t), \tag{4.8}$$

where $\Psi(\mathbf{K}, k) \equiv \Psi(\mathbf{K}, \boldsymbol{\kappa})$ is the spectral function corresponding to the Fourier–Stieltjes components $d\psi(\mathbf{K}, \boldsymbol{\kappa}) \equiv d\psi(\mathbf{K}, k)$:

$$\Psi(\mathbf{K}, k) = \frac{\overline{d\psi(\mathbf{K}, k)\, d\psi^*(\mathbf{K}, k)}}{d\kappa_1\, d\kappa_2}, \tag{4.9}$$

and the function Γ is given by

$$\Gamma(\kappa, k, t) = \left\{\frac{e^{i\sigma} - e^{i\sigma'}}{\sigma - \sigma'} - \frac{e^{i\sigma} - e^{-i\sigma'}}{\sigma + \sigma'}\right\} \left\{\frac{e^{-i\sigma} - e^{-i\sigma'}}{\sigma - \sigma'} - \frac{e^{-i\sigma} - e^{i\sigma'}}{\sigma + \sigma'}\right\}$$

$$= \frac{2}{(\sigma - \sigma')^2}\{1 - \cos(\sigma - \sigma')\} - \frac{2}{\sigma^2 - \sigma'^2}\{1 + \cos 2\sigma' - \cos(\sigma + \sigma') - \cos(\sigma - \sigma')\} + \frac{2}{(\sigma + \sigma')^2}\{1 - \cos(\sigma + \sigma')\}. \tag{4.10}$$

The wave-number k is specified by the incident wave field, and when Γ is regarded as a function of σ', it is evident by inspection that Γ has a maximum when $\sigma = \sigma'$ or $k = \kappa$, when the (scalar) wave-numbers of the incident and scattered waves are equal. Near this maximum, if $\sigma = \sigma' + \epsilon$,

$$\Gamma(\kappa, k, t) \doteqdot \frac{2(1-\cos\epsilon)}{\epsilon^2}, \tag{4.11}$$

neglecting terms of order ϵ/σ. If we consider only the wave components scattered in a given direction θ, the direction of the vector $\boldsymbol{\kappa}$, it follows from (4.8) that the spectral density is greatest near $\kappa = k$, and from (4.10) that the 'band width' $\delta\kappa$ of the spectrum of the scattered components is given by

$$\epsilon = [g(k+\delta\kappa)]^{\frac{1}{2}}t - [gk]^{\frac{1}{2}}t = \pi,$$

or

$$\frac{\delta\kappa}{k} = \frac{2\pi}{t}(gk)^{-\frac{1}{2}} = \frac{2\pi}{nt}. \tag{4.12}$$

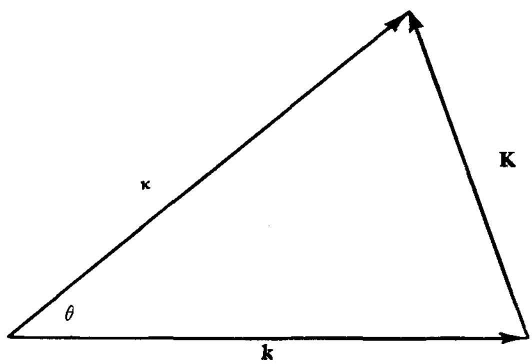

FIGURE 1. The scattering vector.

If the elapsed time t is much greater than the periodic time of the incident waves, this 'band width' of the scattered waves is narrow, and decreases in inverse proportion to t, so that the spectrum of the scattered waves at a given scattering angle θ becomes more and more concentrated about the wave-number $\kappa = k$, the wave-number of the incident waves.

It follows from equation (4.8) in the light of these considerations that the scattered waves of wave-number $\boldsymbol{\kappa} = (\kappa_1, \kappa_2)$ in the direction θ with respect to the direction of the incident waves result from the components of the vorticity field of wave-number $\mathbf{K} = (\kappa_1 - k, \kappa_2)$, where $k = \kappa = (\kappa_1^2 + \kappa_2^2)^{\frac{1}{2}}$. The same vector $\mathbf{K}$ is important in other scattering problems, where it is usually called the 'scattering vector', and the same name can conveniently be used in the present context. It is clear from figure 1 that, since $\kappa = k$, the vector $\mathbf{K}$ is perpendicular to the line bisecting the directions of the incident and scattered waves, and is of magnitude

$$K = 2\kappa \sin\tfrac{1}{2}\theta = 2k\sin\tfrac{1}{2}\theta. \tag{4.13}$$

The appearance of the scattering vector might be expected on geometrical grounds (as shown, for example, by Batchelor 1957) by considering two scattering points P and P', as shown in figure 2, separated by a distance $l = 2\pi/K$.

For reinforcement of the scattered waves, the path APB must equal one wave-length of the incident wave, or $2\pi/k$. Thus $(\pi/k)/(2\pi/K) = \sin\frac{1}{2}\theta$, leading immediately to (4.13).

The mean square amplitude of the waves scattered per unit angle in the direction θ is found from (4.8) by integration over wave-numbers κ in this direction. When the elapsed time t is much greater than the period of the incident waves, the approximation (4.11) can be used, and

$$\Phi(\theta, t) \doteqdot \frac{\alpha^2}{4}\Psi(\mathbf{K})\,t^2\int_0^\infty \frac{2-2\cos\epsilon}{\epsilon^2}\kappa\,d\kappa, \tag{4.14}$$

since over the small range of values of κ corresponding to $-\frac{1}{2}\pi < \epsilon < \frac{1}{2}\pi$, $n' \doteqdot n$ and the variation of the vector K and so of the function $\Psi(K)$ is small. Now,

$$d\epsilon = \tfrac{1}{2}g^{\frac{1}{2}}\kappa^{-\frac{1}{2}}t\,d\kappa \doteqdot \tfrac{1}{2}g^{\frac{1}{2}}k^{-\frac{1}{2}}t\,d\kappa$$

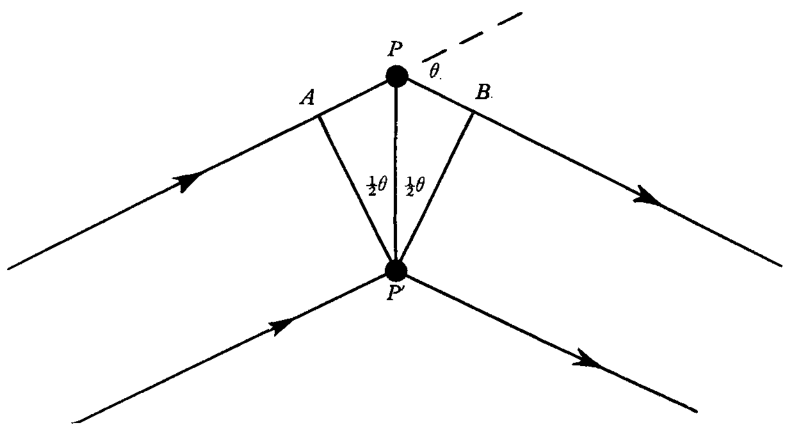

FIGURE 2. A geometrical interpretation of the scattering vector.

near $\kappa = k$, so that (4.14) can be expressed as

$$\Phi(\theta, t) \doteqdot \frac{\alpha^2k^2}{2(gk)^{\frac{1}{2}}}\Psi(\mathbf{K})\,t\int_{-\infty}^{\infty}\frac{1-\cos\epsilon}{\epsilon^2}\,d\epsilon$$

$$= \frac{\pi\alpha^2k^2}{(gk)^{\frac{1}{2}}}\Psi(\mathbf{K})\,t. \tag{4.15}$$

The directional distribution of the scattered waves is described conveniently by the ratio of the mean energy flux developed in the scattered waves per unit length of wave front per unit angle in the direction θ per unit time to the energy flux in the incident wave per unit length of wave front. This ratio is given by

$$s(\theta, k) = \frac{1}{\alpha^2}\frac{\partial}{\partial t}\Phi(\theta, t),$$

$$= \frac{\pi k^2}{(gk)^{\frac{1}{2}}}\Psi(\mathbf{K}). \tag{4.16}$$

which has dimensions $(\text{time})^{-1}$. The fraction of the energy of the incident wave lost by scattering per unit time, or the logarithmic decrement of attenuation due to scattering is

$$\gamma(k) = \int_{-\pi}^{\pi} s(\theta, k)\, d\theta,$$

$$= \frac{\pi k^2}{(gk)^{\frac{1}{2}}} \int_{-\pi}^{\pi} \Psi(\mathbf{K})\, d\theta, \qquad (4.17)$$

where the scattering vector $\mathbf{K}$ has magnitude $2k \sin \frac{1}{2}\theta$ and direction $\frac{1}{2}(\pi + \theta)$.

If further progress beyond expressions such as (4.16) and (4.17) is to be made, some information is necessary concerning the form of the function $\Psi(\mathbf{K})$. It can be seen readily from (4.5) and (4.9) that $\Psi(\mathbf{K})$ has the same dimensions as the two-dimensional vorticity spectrum of the turbulence, and since the Reynolds number of the turbulence encountered in the ocean is extremely large, it is to be expected that $\Psi(\mathbf{K})$ is directly proportional to this vorticity spectrum. The length scales associated with the vorticity spectrum in oceanic turbulence cover a very wide range, perhaps from several hundred metres down to a microscale, or scale of the smallest eddies, of order 1 cm, which certainly includes the wavelengths of incident waves in which we are likely to be most interested. This fact precludes the use of the limiting approximations found useful in considering the scattering of acoustical or radio waves (see Batchelor 1957) which suppose that the length scale of the scatterers is either much greater or much less than the wavelength of the incident waves.

The observation that $\Psi(\mathbf{K})$ is proportional to the two-dimensional vorticity spectrum suggests that use might be made of the predictions of the local similarity theory, but this cannot be done without some caution. We are interested in the form of the vorticity spectrum near the free surface, measured in a plane parallel to the surface. It is well known experimentally, and indeed it is only to be expected on *a priori* grounds that near a *rigid* surface with large mean velocity gradients, the theory of local similarity is inapplicable in a description of the flow. However, a *free* surface imposes fewer constraints on the motion in the immediate neighbourhood; the mean shear now vanishes and the fluid is free to move parallel to the surface. The conditions are much closer to those which the local similarity theory seeks to describe, and the function $\Psi(\mathbf{K})$ to which it might be applied seems to be almost the most appropriate that could be found, since it represents an integral throughout a layer near the surface and should not therefore be crucially dependent upon the conditions immediately at the interface.

With these reservations in mind, then, we will make use of this theory to determine the functional form of $\Psi(\mathbf{K})$. Except when θ is very small (cf. 4.13), the values of K corresponding to waves of moderate length fall in the 'inertial subrange' of the spectrum of the turbulence, so that $\Psi(\mathbf{K})$ should be determined by $\mathbf{K}$ and ϵ, the rate of energy dissipation in the turbulence, and not by any of the other parameters describing the turbulence. On dimensional grounds, therefore,

$$\Psi(\mathbf{K}) = B\epsilon^{\frac{2}{3}} K^{-\frac{2}{3}}, \qquad (4.18)$$

where B is an absolute constant. The directional distribution $s(\theta, k)$ of the scattered waves is found by substitution of (4.18) and (4.13) into (4.17):

$$s(\theta, k) = 2^{-\frac{2}{3}} B\pi g^{-\frac{1}{2}} \epsilon^{\frac{2}{3}} k^{\frac{5}{6}} (\sin \tfrac{1}{2}\theta)^{-\frac{2}{3}}. \qquad (4.19)$$

Near $\theta = 0$, in the forward direction, this expression becomes invalid, since the wave-numqers $K \doteqdot k\theta$ then correspond to length scales in the scattering vorticity field of the same order as the horizontal scale L of the energy-containing components of the turbulence. Indeed, when $\theta = 0$, it can be shown that $s(\theta, k) = 0$, and there is no energy scattered in the direction of the incident wave. This follows from some expressions that we have already derived, for when $\theta = 0$, $K = 0$ and (4.9) and (4.5) can be used to show that $\Psi(0) = 0$, which, in virtue of (4.16) implies that $s(\theta, k) = 0$ when $\theta = 0$. The directional distribution function of the scattered waves is thus of the form shown in figure 3, where the dotted line indicates the continuation of (4.19) near $\theta = 0$. The maximum value of $s(\theta, k)$ occurs at a value of θ of approximately λ/L, where λ is the wavelength of the incident waves.

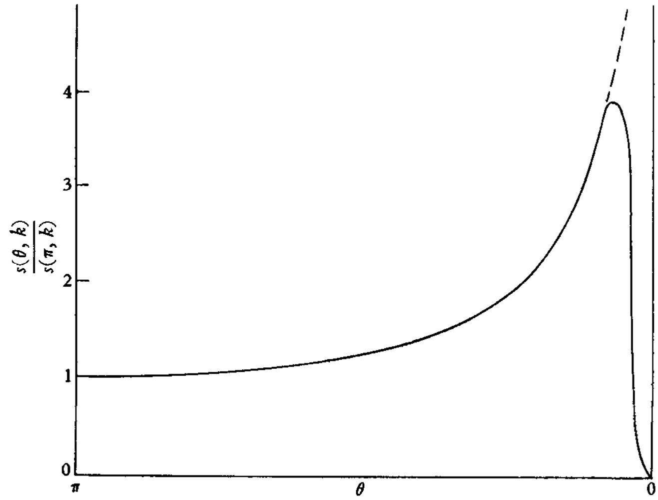

FIGURE 3. The directional distribution of scattered waves in the ocean.

An approximate expression for the logarithmic decrement $\gamma(k)$, valid when λ/L is sufficiently small, is obtained by integration of (4.19) from $\theta = 0$ to 2π and neglecting the decrease of $s(\theta, k)$ to zero when $\theta = 0$. This yields

$$\begin{aligned} \gamma(k) &\doteqdot 2^{-\frac{2}{3}} B\pi g^{-\frac{1}{2}} \epsilon^{\frac{2}{3}} k^{\frac{5}{6}} . 2 \int_0^{\pi} (\sin \tfrac{1}{2}\theta)^{-\frac{2}{3}} \, d\theta \\ &= 2^{\frac{1}{3}} B\pi \left\{ \frac{\Gamma(\frac{1}{6})\, \Gamma(\frac{1}{2})}{\Gamma(\frac{2}{3})} \right\} g^{-\frac{1}{2}} \epsilon^{\frac{2}{3}} k^{\frac{5}{6}} \\ &\doteqdot 29 B g^{-\frac{1}{2}} \epsilon^{\frac{2}{3}} k^{\frac{5}{6}}. \end{aligned} \tag{4.20}$$

It is interesting to compare this result with that derived by Stokes (and discussed by Lamb 1932, §§348, 349) for the logarithmic decrement resulting from viscous dissipation, namely

$$\gamma_v(k) = 2\nu k^2.$$

The attenuation resulting from scattering will predominate if

$$29 B g^{-\frac{1}{2}} \epsilon^{\frac{2}{3}} k^{\frac{5}{6}} > 2\nu k^2,$$

i.e.

$$k < \left\{ \frac{15 B \epsilon^{\frac{2}{3}}}{\nu g^{\frac{1}{2}}} \right\}^{\frac{6}{7}}. \tag{4.21}$$

The rate of turbulent energy dissipation in the open ocean may be estimated from some fairly rough diffusion measurements described by Stommel (1949) and a typical value for ϵ in the open ocean appears to be of order $10^{-5}\,\text{cm}\,\text{sec}^{-3}$. The gravitational acceleration g is approximately $10^3\,\text{cm}\,\text{sec}^{-2}$ and the kinematic viscosity ν for water is approximately $1{\cdot}5\times 10^{-2}\,\text{cm}^2\,\text{sec}^{-1}$. With these values, and if B is of order unity, it follows from (4.21) that attenuation resulting from scattering predominates for wave-numbers less than about $2\times 10^{-2}\,\text{cm}^{-1}$; or for wavelengths greater than about 3 m. In tidal waters where the turbulence may be more intense than it is in the open ocean, the attenuation resulting from scattering may be important for even shorter wavelengths.

It is doubtful whether oceanographical wave measurements are yet sufficiently precise to make meaningful observations on the attenuation of surface waves. The results of this paper, however, suggest that when this is done, the cause of the attenuation of the longer components of the wave field may lie not in the dissipative action of viscosity but in the scattering action of the oceanic turbulence.

This work was supported by the Office of Naval Research under Contract Nonr 38 (248).

REFERENCES

BATCHELOR, G. K. 1957 Article in *Symposium on Naval Hydrodynamics*, 1956. Publication 515, National Academy of Sciences—National Research Council, Washington.

DARBYSHIRE, J. 1952 *Proc. Roy. Soc.* A, **215**, 299.

LAMB, H. 1932 *Hydrodynamics*, 6th ed. Cambridge University Press.

STOMMEL, H. 1949 *J. Mar. Res.* **8**, 199.